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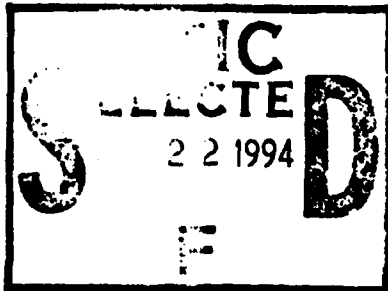
CERTAIN PROBLEMS IN MAGNETOHYDRODYNAMICS

CONSIDERING THE FINITE CONDUCTIVITY OF THE MEDIUM

by I. I. Nochevinka

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FOREWORD

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CERTAIN PROBLEMS IN MAGNETOHYDRODYNAMICS
CONSIDERING THE FINITE CONDUCTIVITY OF THE MEDIUM

An approximating method of finding the parameters of
planar motion of a conducting gas is offered, taking into
account "magnetic" viscosity in the presence of a magnetic
field, perpendicular to the plane of flow.

An approximating technique is advanced for the solution
of the equations which describe the planar, isentropic flow
of an ultrarelativistic gas in an arbitrary magnetic field.

Examinations of the motion of conducting fluids in the
environment of magnetic fields, with the consideration of
magnetic viscosity, have always encountered substantial
mathematical difficulties. Up till now only the special
cases of motion of a fluid with finite conductivity in a
magnetic field have been successfully investigated.

We will examine planar motion of an ideally compres-
sible fluid with finite conductivity in a transverse magnetic

field $H(0, 0, H)$, which can be described by the system of equations

$$\text{rot}\,[\bar{v}^*H^*] = -\gamma_i\nabla^2\bar{H}^*; \quad \text{div}\,\bar{H}^* = 0;$$

$$(\bar{v}^*\nabla)\bar{v}^* = -\frac{1}{\rho^*}\nabla p^* + \frac{f^*}{\rho^*}; \quad \text{div}\,(\rho^*\bar{v}^*) = 0; \quad p^* = f(\rho^*, s^*), \quad (1)$$

where $\gamma_i^* = \frac{c^2}{4\pi\sigma^*}$. – coefficient of magnetic viscosity,

$f^* = -\frac{1}{8\pi}\nabla H^{*2}$ – volume density of the electromagnetic force.

We introduce the dimensionless parameters

$$H = \frac{\bar{H}^*}{a_0\sqrt{\rho_0}}; \quad \bar{v} = \frac{\bar{v}^*}{a_0}; \quad \rho = \frac{\rho^*}{\rho_0}; \quad p = \frac{p^*}{\rho_0 a_0^2}; \quad x = \frac{a_0}{\tau_0}x^*;$$

$$y = \frac{a_0}{\tau_0}y^*; \quad \gamma_i = \frac{\gamma_i^*}{\tau_0}, \quad (2)$$

where by the zero index is signified the corresponding characteristic magnitudes for the given flow. System (1) in dimensionless variables takes the form

$$\text{rot}\,[\bar{v}\bar{H}] = -\gamma_i\nabla^2\bar{H}; \quad \text{div}\,\bar{H} = 0;$$

$$(\bar{v}\nabla)\bar{v} = -\frac{1}{\rho}\nabla P; \quad \text{div}\,(\rho\bar{v}) = 0; \quad P = f_1(\rho, s). \quad (3)$$

where $P = p + \frac{H^2}{8\pi}$ – the total pressure of the gas and of the magnetic field.

In the case of constant finite conductivity $(\eta = \text{const})$, considering that $\text{div}\,\bar{H} = 0$, the equation of induction to within the gradient of an arbitrary function

can be put in the form

$$[\bar{v}\bar{H}] = \tau_i \operatorname{rot} \bar{H},$$ (4)

which is equivalent to the equations

$$v_y H_z = \frac{\partial}{\partial y}(\tau_i H_z); \quad v_x H_z = \frac{\partial}{\partial x}(\tau_i H_z),$$

i. e. $v = \operatorname{grad}\varphi,$ where $\varphi = \tau_i \ln H_z.$ (5)

Thus, the investigation of planar motion of an ideally compressible conducting fluid with a constant coefficient of conductivity in a transverse magnetic field reduces to the investigation, with the aid of well-cultivated methods, of purely hydrodynamical potential flows by the means of the transformed equation of state $[2]$.

In a series of cases it is necessary to consider the variability of the coefficient of conductivity due to the presence of large conductivity gradients. If under these circumstances the conductivity gradient coincides in direction with the conductivity current, as for example in the cooling of the stream in a plasmatron, $[3]$, then $[\operatorname{grad}\tau_i \operatorname{rot} \bar{H}] = 0$ and the equation of induction within the margin of an arbitrary function can be written as

$$[\bar{v}\bar{H}] = \tau_i(x, y) \operatorname{rot} \bar{H}.$$

from which

$$\bar{v}(x, y) = \tau_i(x, y)\operatorname{grad}\varphi,$$ (6)

where $\varphi = \ln H_z$ and $\ln H_z = \operatorname{const}$ form a family of surfaces, normal to the lines of current. This is equivalent to some vortical flow with a coefficient of proportionality

$\eta(x, y)$.

In such cases, when from a series of factors giving rise to a variability in the magnetic viscosity, we can single out a basic one, i.e., we can consider the coefficient of magnetic viscosity as a function of one independent variable, e.g. $\eta = \eta(T)$, the examining of the parameters can be conducted with the assistance of the method developed in work $/\!\_1\_/$.

In the role of an application we will examine the problem of the outflow of an ideally compressible fluid in the presence of a conductivity gradient which coincides in direction with the conductivity current, from an infinite vessel with flat sides in the presence of a perpendicular magnetic field. We will accept the above mentioned suppositions and consider the equation of state in the form $P(\rho, s) = A(s)\rho^2 - B$. Performing calculations analogous to $/\!\_1\_/$, we will obtain equations for the determination of the functions of current ψ and of the quasipotential φ in the variables $r(\eta)$, θ (θ — the angle of inclination of the velocity vector with the x — axis).

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{d}{dr} \ln \left[K \frac{r'(\eta)(6\rho^2 - 9\rho\rho_0 + 4\rho_0^2)}{2\rho^3(\rho_0 - \rho)^2} \right] \frac{\partial \psi}{\partial r} +$$

$$+ \frac{1}{K} \left[- \frac{2\rho^3(\rho_0 - \rho)^2}{r'(\eta)(6\rho^2 - 9\rho\rho_0 + 4\rho_0^2)} \right]^2 \frac{\partial^2 \psi}{\partial \theta^2} = 0. \qquad (7)$$

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{d}{dr} \ln \left[\frac{r'(\eta)(6\rho^2 - 9\rho\rho_0 + 4\rho_0^2)}{2\rho^3(\rho_0 - \rho)^2} \right] \frac{\partial \varphi}{\partial r} +$$

$$+ \frac{1}{K} \left[\frac{2\rho^3(\rho_0 - \rho)^2}{r'(\eta)(6\rho^2 - 9\rho\rho_0 + 4\rho_0^2)} \right]^2 \frac{\partial^2 \varphi}{\partial \theta^2} = 0. \qquad (8)$$

By the introduction of Chapygin's approximating function $K(r)$, equations (7) and (8) can be put in the form $/\!\_1\_/$

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{D_1}{C_1^2}\left(1 - \frac{1}{r^2}\right)\frac{\partial^2 \psi}{\partial w^2} = 0. \qquad (9)$$

where the coefficients D_1 and C_1 are chosen so that they will give the approximate equation of state's best approximation to what is asked for in the defined range of Mach numbers.

Let AB and A'B' be projections of the walls onto the plane XY (Fig. 1). We will turn the X-axis perpendicular to the wall of the vessel through the middle of BB'. We will consider the volume outflow per second of the fluid Q, which by virtue of continuity must be one and the same in all sections of th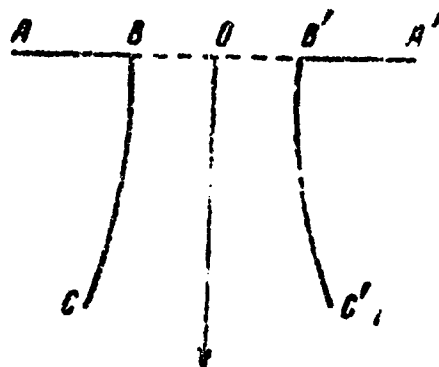e stream. Let ρ_∞ - a plane of fluid at infinity, ρ_1 - a plane outside the vessel, ρ_0 - a plane in an adiabatically restricted gas.

Fig. 1

We will select the line of current along the X-axis as zero, i.e, $\psi(\rho, 0) = 0$ for $0 < \rho_\infty - \rho, \rho_\infty - \rho_1$. By this a constant is immediately defined for each line of current, equal to the outflow of the fluid through a section of a tube of current and passing through the line in question and zero, i.e., $Q = \psi(M_1) - \psi(M_0)$. The boundary conditions $\psi(\rho, \theta)$ for such a choice we obtain as

$$\psi\left(\rho, -\frac{\pi}{2}\right) = \frac{Q}{2} \text{ for } \rho_\infty > \rho > \rho_1; \quad \psi(\rho_\infty - \rho_1, \theta) = \frac{Q}{2} \text{ for } -\frac{\pi}{2} < \theta < 0;$$

$$\psi\left(\rho, \frac{\pi}{2}\right) = -\frac{Q}{2} \text{ for } \rho_\infty > \rho > \rho_1; \quad \psi(\rho_\infty - \rho_1, \theta) =$$

$$= -\frac{Q}{2} \text{ for } 0 < \theta < \frac{\pi}{2}. \qquad (10)$$

Considering the region of variability of the plane of the fluid $r_x > r > r_1$, from the relationship connecting r with $/ 1 /$, we will determine the range of variability of r. Then immediately with the help of (10) we obtain boundary conditions for the function $\psi(r, \theta)$.

$$\psi\left(r, -\frac{\pi}{2}\right) \quad \frac{Q}{2} \quad \text{for } r_x > r > r_1;$$

$$\psi(r_x - r_1, \theta) = \frac{Q}{2} \qquad \text{for } -\frac{\pi}{2} \quad \theta \quad 0;$$

$$\psi\left(r, +\frac{\pi}{2}\right) \quad -\frac{Q}{2} \qquad \text{for } r_x \quad r > r_1;$$

$$\psi \quad (r_x - r_1, \theta) \quad \frac{Q}{2} \quad \text{for } 0 < \theta \quad \frac{\pi}{2} . \tag{11}$$

Equation (9) can be solved by a Fourier method

$$\psi(r, \theta) \quad R(r) \, \Phi(\theta). \tag{12}$$

For the functions R(r) and $\Phi(\theta)$ we obtain the equations

$$\frac{d^2 \Phi(\theta)}{d\theta^2} \quad \mu^2 \Phi(\theta) \quad 0. \tag{13}$$

$$\frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} \quad \mu^2 A^* \left(\frac{1}{r^2} - 1\right) R(r) \quad 0. \tag{14}$$

where

$$A^* \quad \frac{D_i}{C_i^2} \quad 0.$$

From the condition of periodicity for the function $\Phi(\theta)$

$$\Phi(\psi) = \Phi(\psi + 2\pi), \quad \Phi'(\psi) = \Phi'(\psi + 2\pi) \tag{15}$$

it follows that $\mu^2 = n^2$, where n is an integer. Introducing a new variable $\bar{r} = -\sqrt{A^*}\, nr = \nu r$ into Bessel's equation (14) we will obtain

$$\frac{d^2 R(\bar{r})}{d\bar{r}^2} + \frac{1}{\bar{r}} \frac{dR(\bar{r})}{d\bar{r}} - \left(1 - \frac{1}{\bar{r}^2}\right) R(\bar{r}) = 0, \tag{16}$$

whose solution is expressible by Bessel functions of imaginary argument.

$$R(\bar{r}) = C_1' I_1 + C_2' K_1. \tag{17}$$

From the conditions of boundedness of the fluid plane at infinity $\rho_\infty < N$ the boundedness of $\bar{r}(\eta)$ follows; from where $C_1' = 0$. The general solution of equation (14) we obtain in the form

$$\psi(r, \psi) = \sum_n [A_{1n} \cos(n\psi) + A_{2n} \sin(n\psi)] K_1^{(\nu)} (\sqrt{-A^*}\, nr). \tag{18}$$

The function $\varphi(r, \theta)$ can be analogously defined.

The transition to the physical plane can be accomplished with the aid of the formulas

$$dx = \frac{1}{v} \left[\left(\eta \cos\theta \frac{\partial \bar{r}}{\partial r} - \frac{1}{\rho} \sin\theta \frac{\partial \psi}{\partial r} \right) dr + \right.$$

$$\left(\tau_1 \cos \psi \frac{\partial z}{\partial M} - \frac{1}{\rho} \sin \psi \frac{\partial z}{\partial M} \right) d\psi \bigg|,$$

$$dy = \frac{1}{v} \left[\left(\tau_1 \sin \psi \frac{\partial z}{\partial r} + \frac{1}{\rho} \cos \psi \frac{\partial z}{\partial r} \right) dr + \right.$$

$$\left. + \left(\tau_1 \sin \psi \frac{\partial z}{\partial M} - \frac{1}{\rho} \cos \psi \frac{\partial z}{\partial M} \right) d\psi \right].$$

(19)

Thus the methods developed in hydrodynamics can be effectively applied not only in the investigation of planar motion of a conducting fluid in a transverse magnetic field, when the conductivity is infinitely large, but also in a series of cases when it is necessary to concern oneself with the presence of magnetic viscosity, constant or variable with the gradient which coincides with the direction of the conductivity current.

The approximating method described can also be applied in the examination of planar problems in the ultrarelativistic case of motion of a conducting medium in the presence of arbitrary magnetic fields.

The motion of a relitivistic gas in the presence of arbitrary magnetic fields can be represented in the form of an equality to zero of the divergence of the total mechanical and electromagnetic tensor of the energy-impulse. In the case of an infinite conductivity of the gas, the energy-impulse tensor can be written in the form

$$T_{ik} = \frac{w^*}{V} U_i U_k + p^* \delta_{ik} \quad (i, k = 1, 2, 3, 4).$$

(20)

where $w^* = pV + \rho V c^2 + w'$ - the total heat content of the gas and of the magnetic field, $p^* = p + p'$ - the total pressure, V - the specific volume, U_i - 4-velocity, $x_{1,2,3} = x, y, z; x_4 = ict$.

It can be proved that in the case of arbitrarily

chosen magnetic fields it is sufficient to limit oneself
only to the investigation of the modified tensor of
energy-impulse of macroscopic bodies, which include in
themselves additional heat content w' and additional
pressure p' . In the general case w' and p' can
easily be calculated by the formulas

$$w' = \frac{V}{4\pi}\left(F_{il}F_{kl} + \frac{u^2}{c^2}\frac{F_{lm}^2}{4\theta^2}\delta_{ik}\right)U_iU_k; \quad p' = -\frac{F_{lm}^2}{16\pi\theta^2}\quad \left(\theta = \sqrt{1-\frac{u^2}{c^2}}\right), \quad (21)$$

where F_{ik} — components of the electromagnetic field
tensor in the system of representation K' , relative to
which the given element of gas travels with velocity u .
The expediency of knowing the total energy-impulse tensor
of the system, gas and electromagnetic field, compared to
the tensor of the macroscopic bodies appears during the
transition from one representation to the other, seeing
that the Lorentz transformation for the components of the
tensor of an electromagnetic field F_{ik} is much simpler
than for the components of the electromagnetic tensor of
energy-impulse T_{ik}.

An investigation of the parameters of planar flow of
an ultrarelativistic gas in a given magnetic field in the
case of maintaining isentropic behaviour can be conducted
with the aid of the method set forth in work $/1/$. We
will conduct an investigation for an infinitesimally
small element of gas in the laboratory system K' . In
the case of isentropic flows, there exists, as is known,
a relativistic analogue of the potential

$$\frac{\partial\varphi}{\partial x_i} = U_i w^*, \quad (22)$$

from which for planar flows we have for i 1.2

$$w^*U_1 = \frac{\partial\varphi}{\partial x_1} ; \quad w^*U_2 = \frac{\partial\varphi}{\partial x_2} . \quad (23)$$

Introducing also with the help of the relativistic equation of continuity the relativistic analogue of the function of current

$$\frac{U_1}{V} = \frac{\partial \psi}{\partial x_2} ; \quad \frac{U_2}{V} = -\frac{\partial \psi}{\partial x_1} , \qquad (24)$$

we obtain equations analogous to the equations for stationary, planar, vortical flow of a common gas $\diagup 1 \diagdown$

$$\frac{1}{w^*} \frac{\partial \varphi}{\partial x} = V \frac{\partial \psi}{\partial y} ; \quad \frac{1}{w^*} \frac{\partial \varphi}{\partial y} = -V \frac{\partial \psi}{\partial x} \; (x_{1,2} = x, y), \qquad (25)$$

where the role of the coefficient of proportionality is filled by the magnitude of the reciprocal of the total heat content. Thus, the problem reduces to the solution of the equations (25) with the use of the relativistic equation of Bernoulli

$$\frac{w^*}{h} = w_0^* = \text{const} \qquad (26)$$

and of the equation of state

$$p^* = (\gamma - 1) z^* . \; (1 \quad \gamma \quad 2), \qquad (27)$$

which in the case of an ultrarelativistic system is wholly determined by the fixing of one thermodynamic function (e.g. w^*). Introducing new independent variables $\xi \; (w^*)$ and ψ, where u – angle, formed by the 3-vector of velocity u with the X-axis, and noting that $u = u(w^*)$, $V = V(w^*)$, we can put the equations (25) in the form

$$\frac{1}{w^*} \frac{\partial \varphi}{\partial \xi} = \left(\frac{\partial V}{\partial w^*} - \frac{V}{u} \frac{\partial u}{\partial w} \right) \frac{dw^*}{d\xi} \frac{\partial \psi}{\partial x} ; \quad \frac{1}{w^*} \left(\frac{1}{w^*} - \frac{1}{u} \frac{\partial u}{\partial w^*} \right) \frac{dw^*}{d\xi} \frac{\partial \varphi}{\partial x} =$$
$$= V \frac{\partial \psi}{\partial \xi} . \qquad (28)$$

With this we suppose that the functions ψ and φ are continuous, finite, single-valued and that the Jacobian

$\left(\dfrac{x,y}{\xi,z}\right) \neq 0$ in all regions of the flow. Placing in (28) the quantities $u(w^*)$ and $\dfrac{\partial u}{\partial w^*}$, defined in (26), $V'(w^*)$ and

$\dfrac{\partial V}{\partial \cdot}$ in (27), noting in the limiting case $\gamma = \dfrac{4}{3}$, and imposing on ξ (arbitrary function of w^*) the simplified condition

$$\frac{A_1(-3w^{*-2} + 4)}{w^*(1 - w^{*2})} \frac{dw^*}{d\xi} = 1.$$

(29)

we put equations (28) in the form

$$\frac{\partial \varphi}{\partial \xi} = \frac{\partial \psi}{\partial z} ; \quad \frac{\partial \varphi}{\partial z} = -K(w^*) \frac{\partial \psi}{\partial \xi} ,$$

(30)

where

$$K(w^*) = \frac{A_1(-3w^{*2} + 4)}{w^{*2}(1 - 2w^{*2})}$$

is the Chapygin function introduced by us.

The solution of equations (30) with the help of Chapygin's approximating method is put forth in work $\angle 1\angle$.

Isentropic flow of an ultrarelativistic gas in a perpendicular magnetic field is examined by the surmised method. By this it comes to light that a perpendicular magnetic field does not violate the isentropicity of an ultrarelativistic gas. In the case of a magnetic

field arbitrarily disposed in the plane of flow, isentropic behaviour is generally violated.

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